

ON SOME GEOMETRIC REPRESENTATIONS OF $\mathrm{GL}_n(\mathfrak{o})$

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ABSTRACT. We study a family of complex representations of the group $\mathrm{GL}_n(\mathfrak{o})$, where \mathfrak{o} is the ring of integers of a non-archimedean local field F . These representations occur in the restriction of the Grassmann representation of $\mathrm{GL}_n(F)$ to its maximal compact subgroup $\mathrm{GL}_n(\mathfrak{o})$. We compute explicitly the transition matrix between a geometric basis of the Hecke algebra associated with the representation and an algebraic basis which consists of its minimal idempotents. The transition matrix involves combinatorial invariants of lattices of submodules of finite \mathfrak{o} -modules. The idempotents are p -adic analogs of the multivariable Jacobi polynomials.

1. INTRODUCTION

1.1. Outline. Let F be a non-archimedean local field and \mathfrak{o} its ring of integers. Let k_F be the residue field and \mathfrak{p} the maximal ideal of \mathfrak{o} . For $\ell \in \mathbb{N}$, let \mathfrak{o}_ℓ denote the finite quotient $\mathfrak{o}/\mathfrak{p}^\ell$. The group $\mathrm{GL}_n(F)$ acts on $\mathrm{Gr}(m, n, F)$, the Grassmannian of m -dimensional subspaces of a fixed n -dimensional space, giving rise to a complex representation of $\mathrm{GL}_n(F)$ on $L^2(\mathrm{Gr}(m, n, F))$. This work is about the restriction of this representation to the maximal compact subgroup $\mathrm{GL}_n(\mathfrak{o})$. Since the irreducible constituents of this representation are contained in $\mathcal{S}(\mathrm{Gr}(m, n, F)) \subset L^2(\mathrm{Gr}(m, n, F))$, the dense subspace of locally constant functions, we focus on the latter and call it the *Grassmann representation* of $\mathrm{GL}_n(\mathfrak{o})$. The Grassmann representation has a multiplicity free decomposition to irreducible representations

$$(1) \quad \mathcal{S}(\mathrm{Gr}(m, n, F)) = \bigoplus_{\lambda \in \Lambda_m} \mathcal{U}_\lambda^F,$$

where Λ_m stands for partitions of at most m parts, see [BO]. Each irreducible constituent \mathcal{U}_λ^F contains a unique (normalized) P_x -spherical vector e_λ^F , where P_x is a stabilizer of a point $x \in \mathrm{Gr}(m, n, F)$. These functions are the non-archimedean analogs of the multivariable Jacobi polynomials which arise in the analogous setup when F is either \mathbb{R} or \mathbb{C} , and will therefore be called the p -adic multivariable Jacobi functions. Algebraically, after the appropriate normalization, they form a basis of the Hecke algebra $\mathcal{H}_m = \mathrm{End}_{\mathrm{GL}_n(\mathfrak{o})}(\mathcal{S}(\mathrm{Gr}(m, n, F)))$ which consists of minimal idempotents. The algebra \mathcal{H}_m can be identified with the convolution algebra $\mathcal{S}(P_x \backslash \mathrm{GL}_n(\mathfrak{o}) / P_x)$ of bi- P_x -invariant locally constant functions on $\mathrm{GL}_n(\mathfrak{o})$. The latter description comes with a natural geometric basis: characteristic functions of the

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double cosets. The main result in this paper is an explicit computation of the transition matrix between these geometric and algebraic bases (Theorem 7).

The Grassmann representation is filtered by the finite dimensional I_ℓ -invariant subspaces

$$(2) \quad (0) \subset \mathcal{S}(\mathrm{Gr}(m, n, F))^{I_1} \subset \cdots \subset \mathcal{S}(\mathrm{Gr}(m, n, F))^{I_\ell} \subset \cdots \subset \mathcal{S}(\mathrm{Gr}(m, n, F)),$$

where $I_\ell = \mathrm{Ker}\{\mathrm{GL}_n(\mathfrak{o}) \rightarrow \mathrm{GL}_n(\mathfrak{o}_\ell)\}$. In fact, $\mathcal{S}(\mathrm{Gr}(m, n, F)) = \varinjlim \mathcal{S}(\mathrm{Gr}(m, n, F))^{I_\ell}$, hence the problem can be translated into a finite problem: analysis of the representation of the finite group $\mathrm{GL}_n(\mathfrak{o}_\ell)$ in $\mathcal{S}(\mathrm{Gr}(m, n, F))^{I_\ell}$. The latter, can in turn be identified with $\mathbb{C}(\mathrm{Gr}(m, n, \mathfrak{o}_\ell))$, the space of complex valued functions on the (finite) Grassmannian of free \mathfrak{o}_ℓ -submodules of rank m in \mathfrak{o}_ℓ^n , with its natural $\mathrm{GL}_n(\mathfrak{o}_\ell)$ -action.

1.2. Context of the problem. To put things into perspective, we briefly describe the archimedean [JC] and quantum [DS] counterparts of the Grassmann representation, see also [Onn, OS] for more details. These are representations of the orthogonal group O_n arising from its action on $\mathrm{Gr}(m, n, \mathbb{R})$, of the unitary group U_n arising from its action on $\mathrm{Gr}(m, n, \mathbb{C})$, and of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ on the quantum Grassmanian [DS]. They all have similar decompositions to (1), indexed by the same set Λ_m , giving rise to zonal spherical functions $e_\lambda^\mathbb{R}$ and $e_\lambda^\mathbb{C}$, which are special cases of the multivariable Jacobi polynomials, and e_λ^q which are known as the multivariable little q -Jacobi polynomials [Sto].

In [Sto, SK] it is shown that by taking appropriate limits, the multivariable q -Jacobi polynomials e_λ^q degenerate to the multivariable Jacobi polynomials which specialize to $e_\lambda^\mathbb{R}$ and $e_\lambda^\mathbb{C}$, and in [Onn] it is further shown that they degenerate to the p -adic multivariable Jacobi functions e_λ^F which are studied in the present paper. Besides the esthetic nature of these limits, in which the quantum zonal spherical functions degenerate to the zonal spherical functions over all local fields, archimedean and non-archimedean, they have been used in [OS] to compute the dimensions of the irreducible constituents in (1), and no other direct computation is known at present.

In a different direction, the multivariable p -adic Jacobi functions generalize the q -Hahn polynomials for $q = |\mathfrak{o}/\mathfrak{p}| = |k_F|$ (see e.g. [GR] for the precise definition), which under the appropriate interpretation coincide with e_λ^F , for $\lambda = (1^j)$ with $0 \leq j \leq m$. These functions form a basis of $\mathrm{End}_{\mathrm{GL}_n(k_F)} \mathbb{C}(\mathrm{Gr}(m, n, k_F))$, which as was mentioned above, captures the first term in the filtration (2). See [Del, Dun] for more details on this special case.

The irreducible representations \mathcal{U}_λ^F are studied in [BO] in great detail. Their precise identification involves the study a wider family of geometric representations of the group $\mathrm{GL}_n(\mathfrak{o}_\ell)$ which arise from its action on $\mathrm{Gr}(\lambda, \mathfrak{o}_\ell^n)$, the Grassmannian of submodules of \mathfrak{o}_ℓ^n of type λ . Isomorphism types of submodules of \mathfrak{o}_ℓ^n are classified by partitions, which is the underlying reason behind the appearance of Λ_m in (1).

1.3. Content of the paper. Section 2 is devoted to representations and Hecke algebra which arise from the action of the finite quotients $\mathrm{GL}_n(\mathfrak{o}_\ell)$ on Grassmannians of submodules of \mathfrak{o}_ℓ^n . The main tools, i.e. geometrically defined intertwining operators, are described and developed. Most of this section is an adaptation of relevant results and ideas from [BO] which is the foundational background for this work.

Sections 3 and 4 are devoted to transition matrices between various bases of the Hecke algebras. The main tools which are used are combinatorial invariants of the lattice of submodules in free \mathfrak{o}_ℓ -modules, the Euler characteristic of the simplicial complex associated with flags of submodules, and explicit computations with q -binomial coefficients.

In section 5 the finite results are transferred to $\mathcal{S}(\text{Gr}(m, n, F))$ and its Hecke algebra \mathcal{H}_m . Special attention is given to the limiting process from the algebraic and topological aspects.

Section 6 is devoted to related topics and open problems. In the appendix several claims on modules over discrete valuation rings are proved, which we suspect to be known, but could not find an adequate reference.

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2. HECKE ALGEBRAS ASSOCIATED TO FINITE GRASSMANNIANS

Let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$ and let G_{ℓ^n} denote the automorphism group of a free \mathfrak{o}_ℓ -module of rank n . Upon a choice of a basis G_{ℓ^n} can be identified with $\text{GL}_n(\mathfrak{o}_\ell)$. Recall that any finitely generated \mathfrak{o}_ℓ -module is isomorphic to $\mathfrak{o}_\lambda = \bigoplus_{i=1}^r \mathfrak{o}/\mathfrak{p}^{\lambda_i}$ for some $\lambda = (\lambda_i)$ where $\ell \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0$ since \mathfrak{o} is a discrete valuation ring. We call λ the (isomorphism) *type* of that module. The *length* of the partition λ , i.e. the number of its nonzero parts, is the rank of the module \mathfrak{o}_λ , and the *height* of a partition is its largest part. The set of partitions of length at most m is denoted Λ_m . The set of types is equipped with a natural partial order: $\mu \leq \nu$ whenever a module of type μ can be embedded in a module of type ν . In terms of the corresponding Young diagrams it amounts to inclusion of the corresponding diagrams. Here and in the sequel we follow the notations of [Mac, II.1].

In this section we analyze representations and Hecke algebras which arise from Grassmannians of submodules of the free \mathfrak{o}_ℓ -module of rank n . On one hand they generalize the case $\ell = 1$ which is studied in [Dun], and on the other hand they form the crucial ingredient for understanding the Grassmann representation $\mathcal{S}(\text{Gr}(m, n, F))$ of $\text{GL}_n(\mathfrak{o})$.

2.1. Definition and realization of the Hecke algebra. Let $\mathcal{L}(\mathfrak{o}_\ell^n)$ denote the lattice of submodules of \mathfrak{o}_ℓ^n . The group G_{ℓ^n} acts on the lattice $\mathcal{L}(\mathfrak{o}_\ell^n)$ which is a disjoint union of the G_{ℓ^n} -invariant subsets

$$X_\lambda = \text{Gr}(\lambda, \mathfrak{o}_\ell^n) = \{x \in \mathcal{L}(\mathfrak{o}_\ell^n) \mid x \simeq \mathfrak{o}_\lambda\} \quad (\lambda \in \Lambda_n^\ell),$$

where $\Lambda_n^\ell = \{\lambda \in \Lambda_n \mid \text{height}(\lambda) \leq \ell\}$ stands for isomorphism types of elements in $\mathcal{L}(\mathfrak{o}_\ell^n)$. Let $\tau : \mathcal{L}(\mathfrak{o}_\ell^n) \rightarrow \Lambda_n^\ell$ be the *type map* which assigns to each module its isomorphism type.

For each $\lambda \in \Lambda_n^\ell$ define a representation of G_{ℓ^n} arising from this action on $\mathcal{F}_\lambda = \mathbb{C}(X_\lambda)$, the vector space of \mathbb{C} -valued functions on X_λ , with inner product induced from the counting measure. The following claim, which will be proved in §5, highlights the relevance of the representations \mathcal{F}_λ with $\lambda = (\ell^m) = (\ell, \dots, \ell)$.

Claim 2.1. *There exist an isomorphism of $\text{GL}_n(\mathfrak{o})$ -representations*

$$\mathcal{S}(\text{Gr}(m, n, F)) \simeq \varinjlim \mathcal{F}_{\ell^m}.$$

Apart from their role in Claim 2.1, the representations \mathcal{F}_{ℓ^m} are distinguished among all \mathcal{F}_λ since

- (a) They are multiplicity free, and
- (b) The number of their irreducible constituents is $|\Lambda_m^\ell| = \binom{\ell+m}{m}$, in particular, it is independent of \mathfrak{o} .

This is proved in [BO] in greater generality, but in order to be as self contained as possible we shall explain it in detail. Let

$$\mathcal{H}_{\ell^m} = \text{End}_{G_{\ell^n}}(\mathcal{F}_{\ell^m})$$

stand for the (Hecke) algebra of G_{ℓ^n} -invariant endomorphisms of \mathcal{F}_{ℓ^m} . Assertions (a) and (b) would follow once we show that the algebra \mathcal{H}_{ℓ^m} is isomorphic to the algebra $\mathbb{C}^{\Lambda_m^\ell}$ of complex valued functions on Λ_m^ℓ with pointwise multiplication. To prove that, we look at an alternative description of \mathcal{H}_{ℓ^m} which is of geometric flavor. The algebra $\text{End}_{\mathbb{C}}(\mathcal{F}_{\ell^m})$ can be identified with $\mathbb{C}(X_{\ell^m} \times X_{\ell^m})$ by interpreting any function $f : X_{\ell^m} \times X_{\ell^m} \rightarrow \mathbb{C}$ as a summation kernel $T_f : \mathcal{F}_{\ell^m} \rightarrow \mathcal{F}_{\ell^m}$, that is, $T_f(h)(x) = \sum_{y \in X_{\ell^m}} f(x, y)h(y)$. The map $f \mapsto T_f$ is just the identification of matrices with endomorphisms, which is also equivariant with respect to the natural G_{ℓ^n} -action on both algebras: $[g \cdot T](h) = g(T(g^{-1}h))$ and $[g \cdot f](x, y) = f(gx, gy)$ for $g \in G_{\ell^n}$, $T \in \mathcal{F}_{\ell^m}$, and $f \in \mathbb{C}(X_{\ell^m} \times X_{\ell^m})$. Taking G_{ℓ^n} -invariants gives

$$(3) \quad \mathcal{H}_{\ell^m} \simeq \mathbb{C}(X_{\ell^m} \times_{G_{\ell^n}} X_{\ell^m}).$$

Claim 2.2. *For $m \leq n/2$ there exist a bijection*

$$\begin{aligned} X_{\ell^m} \times_{G_{\ell^n}} X_{\ell^m} &\xrightarrow{\sim} \Lambda_m^\ell \\ G_{\ell^n}(x, y) &\longmapsto \tau(x \cap y), \end{aligned}$$

Proof. The fact that G_{ℓ^n} preserves the module structure implies that this map is well defined. It is onto due to the assumption $m \leq n/2$ which gives enough room to realize any type λ as intersection of two \mathfrak{o}_ℓ -modules of type ℓ^m . It is one-to-one because any abstract isomorphism between $x \cap y$ and $x' \cap y'$ can be lifted to an element $g \in G_{\ell^n}$ such that $(x', y') = (gx, gy)$, using

Lemma 2.3. *Let $z \subset E$ and $z' \subset E'$ be modules such that $z \simeq z'$ and $E \simeq E' \simeq \mathfrak{o}_\ell^j$. Then any isomorphism of \mathfrak{o}_ℓ -modules $z \rightarrow z'$ can be extended to an isomorphism $E \rightarrow E'$.*

Proof. The ring \mathfrak{o}_ℓ is self injective¹, therefore, the module \mathfrak{o}_ℓ^j is injective, and hence the embedding $z \xrightarrow{\sim} z' \hookrightarrow E'$ can be extended to a map $E \rightarrow E'$. It is easy to see that among such extensions exist one-to-one extensions which must be surjective as well due to the finiteness of \mathfrak{o}_ℓ . \square

Going back to the argument above, the isomorphism $x \cap y \simeq x' \cap y'$ can be (simultaneously) extended to isomorphisms $x \simeq x'$ and $y \simeq y'$ by using the lemma for $j = m$. These two isomorphisms glue to an isomorphism $x + y \simeq x' + y'$, which by using the lemma once more for $j = n$, proves the claim. \square

Corollary 1. *The algebra \mathcal{H}_{ℓ^m} is semisimple, commutative and of dimension $|\Lambda_m^\ell|$, hence*

$$(4) \quad \mathcal{H}_{\ell^m} \simeq \mathbb{C}^{\Lambda_m^\ell}.$$

Proof. Given the bijection of Claim 2.2 we get that both \mathcal{H}_{ℓ^m} and $\mathbb{C}^{\Lambda_m^\ell}$ have the same dimension: $|\Lambda_m^\ell| = |\{\text{isomorphism types of submodules of } \mathfrak{o}_\ell^m\}|$. Both are semisimple, thus the only non obvious issue is the commutativity of the algebra \mathcal{H}_{ℓ^m} , which follows from Gelfand's trick. That is, we have that (x, y) and (y, x) are in the same G_{ℓ^n} -orbit since $\tau(x \cap y) = \tau(y \cap x)$, hence the identity is an anti-isomorphism of the algebra. \square

¹This is well known, but can also be easily verified using Baer's criterion which reduces the injectivity verification to a trivial calculation.

It will be convenient to denote from now on $\phi = \ell^m$ and $\Phi = \ell^n$. To make the link with the terminology of [BO], note that Lemma 2.3 shows that rectangular types such as ϕ and Φ are symmetric (Definition 2.1 in *loc. cit.*), and Claim 2.2 shows that (ϕ, Φ) form a symmetric couple (Definition 2.2 in *loc. cit.*), provided that $m \leq n/2$, which is our assumption throughout.

Using the isomorphisms (3) and (4), we obtain two natural bases for \mathcal{H}_ϕ . The first, which comes from the r.h.s of (3), consists of the operators corresponding to characteristic functions of the orbits $\{\mathbf{g}_\lambda \mid \lambda \leq \phi\}$, and will be called the *geometric basis*, and the second, which comes from the r.h.s of (4), consists of idempotents $\{\mathbf{e}_\lambda \mid \lambda \leq \phi\}$, and will be called the *algebraic basis*. The precise meaning of the indexing of the algebraic basis will follow from Theorem 2 below. In order to connect these bases we need a fine analysis of intertwining operators which is the theme of the next subsection.

2.2. Geometric intertwiners and bases for the Hecke algebra. Define the following operators

(a) For each pair of types $\lambda \leq \mu$ let

$$\begin{aligned} T_{\mu \succ \lambda} : \mathcal{F}_\lambda &\rightarrow \mathcal{F}_\mu, & T_{\mu \succ \lambda} h(y) &= \sum_{x \subset y} h(x) & (y \in X_\mu) \\ T_{\lambda \prec \mu} : \mathcal{F}_\mu &\rightarrow \mathcal{F}_\lambda, & T_{\lambda \prec \mu} h(x) &= \sum_{y \supset x} h(y) & (x \in X_\lambda). \end{aligned}$$

(b) For types $\lambda, \mu \leq \nu$, let $T_{\lambda \hookleftarrow \nu \hookrightarrow \mu} : \mathcal{F}_\mu \rightarrow \mathcal{F}_\lambda$ be the operator

$$T_{\lambda \hookleftarrow \nu \hookrightarrow \mu} h(x) = \sum_{\{y \in X_\mu \mid \tau(y+x) = \nu\}} h(y) \quad (x \in X_\lambda).$$

(c) The aforementioned operators $\mathbf{g}_\lambda \in \mathcal{H}_\phi$ ($\lambda \leq \phi$) are explicitly defined by

$$\mathbf{g}_\lambda h(x) = \sum_{\{y \mid \tau(y \cap x) = \lambda\}} h(y) \quad (x \in X_\phi).$$

Note that all these operators commute with the G_Φ -action, and that $T_{\mu \succ \lambda}$ and $T_{\lambda \prec \mu}$ form an adjoint pair. In order to minimize confusion, we follow the rule that whenever an operator is labeled with a diagram (e.g. $T_{\lambda \hookleftarrow \nu \hookrightarrow \mu}$), it acts from the space indexed by the *right* type of the diagram (\mathcal{F}_μ) to the space indexed by the *left* type (\mathcal{F}_λ).

In order to find the transition matrix between the geometric basis $\{\mathbf{g}_\lambda\}$ and the algebraic basis $\{\mathbf{e}_\lambda\}$ of \mathcal{H}_ϕ , we introduce a third basis which is defined by $\mathbf{c}_\lambda = T_{\phi \succ \lambda} T_{\lambda \prec \phi}$ ($\lambda \leq \phi$), and will be called the *cellular basis*. The geometric basis can be viewed as averaging operators along ‘spheres’, whereas the cellular basis can be viewed as weighted averaging operators on ‘balls’. More specifically, the following upper triangular relation holds [BO, §3.4.2]

$$(\mathbf{c}\text{-}\mathbf{g}) \quad \mathbf{c}_\lambda = \sum_{\phi \geq \nu \geq \lambda} \binom{\nu}{\lambda} \mathbf{g}_\nu,$$

which in turn proves that $\{\mathbf{c}_\lambda\}$ is indeed a basis. Here $\binom{\nu}{\lambda}$ is the number of submodules of type λ contained in a module of type ν . For each $\lambda \leq \phi$ set

$$\mathcal{H}_\phi^\lambda = \text{Span}_{\mathbb{C}}\{\mathbf{c}_{\lambda'} \mid \lambda' \leq \lambda\}, \quad \text{and} \quad \mathcal{H}_\phi^{\lambda^-} = \text{Span}_{\mathbb{C}}\{\mathbf{c}_{\lambda'} \mid \lambda' < \lambda\}.$$

The following is proved in [Hil], and in a greater generality in [BO].

Theorem 2. \mathcal{H}_ϕ^λ and $\mathcal{H}_\phi^{\lambda-}$ are ideals $\forall \lambda \leq \phi$, hence, $\{\mathcal{K}_\lambda = \mathcal{H}_\phi^\lambda / \mathcal{H}_\phi^{\lambda-}\}_{\lambda \leq \phi}$ is a complete set of irreducible representations of \mathcal{H}_ϕ .

Corollary 3. If \mathbf{e}_λ is the idempotent in \mathcal{H}_ϕ corresponding to \mathcal{K}_λ for all $\lambda \leq \phi$, then there exist a lower triangular matrix $(A_{\lambda\kappa})$ such that

$$(c-e) \quad \mathbf{c}_\lambda = \sum_{\kappa \leq \lambda} A_{\lambda\kappa} \mathbf{e}_\kappa.$$

The cellular basis appears as a bridge between the geometric and algebraic bases. It is upper triangular with respect to the former and lower triangular with respect to the latter. In the next subsection we shall use it to compute the idempotents explicitly.

Remark 1. Theorem 2 can be used to label the irreducible representations of \mathcal{H}_ϕ . The one-dimensional \mathcal{H}_ϕ -module \mathcal{K}_λ is the unique \mathcal{H}_ϕ -module which is annihilated by all $\{\mathcal{H}_\phi^\mu \mid \mu < \lambda\}$ and not annihilated by \mathcal{H}_ϕ^λ . In view of the well known dictionary between representations of the group which occur in \mathcal{F}_ϕ and modules of the Hecke algebra \mathcal{H}_ϕ , we can now label the irreducibles in \mathcal{F}_ϕ by: $\mathcal{U}_\lambda \leftrightarrow \mathcal{K}_\lambda$. Moreover, by the definition of \mathbf{c}_λ as the composition $T_{\phi \succ \lambda} T_{\lambda \prec \phi}$, the annihilation criterion above translates to the fact that \mathcal{U}_λ occurs in \mathcal{F}_λ and does not occur in \mathcal{F}_μ for $\mu < \lambda$.

3. TRANSITION MATRIX: CELLULAR TO GEOMETRIC

In this section we invert the relation **(c-g)** and compute it explicitly. It consists of two parts, an abstract inversion using properties of the lattice of submodules and an explicit calculation in terms of q -binomial coefficients, where q is the cardinality of the residue field k_F .

3.1. An abstract inversion. Two \mathfrak{o} -module monomorphisms $i : x \hookrightarrow y$ and $i' : x' \hookrightarrow y'$ are said to be equivalent if there are isomorphisms $x \simeq x'$ and $y \simeq y'$ such that the following diagram is commutative

$$(5) \quad \begin{array}{ccc} x & \xhookrightarrow{i} & y \\ \downarrow & & \downarrow \\ x' & \xhookrightarrow{i'} & y' \end{array}$$

Assuming the isomorphism types of x and y are λ and ν correspondingly, we denote the equivalence class of $i : x \hookrightarrow y$ by $i : \lambda \hookrightarrow \nu$, and let $\binom{\nu}{i : \lambda \hookrightarrow \nu}$ stand for the number of submodules of type λ in a module of type ν with embedding type i .

Let y be a finite module over a \mathfrak{o} . Denote by $\mathcal{L}(y)$ the lattice of submodules of y . One naturally associates a simplicial complex to y , denoted $C(y)$, with vertices consisting of the non-trivial submodules of y (all but 0 and y). The simplices of $C(y)$ are given by flags

$$\{(y_1, y_2, \dots, y_m) \mid 0 \subset y_1 \subset y_2 \subset \dots \subset y_m \subset y\}$$

We denote the Euler characteristic of $C(y)$ by $\chi(y)$.

For inverting the relation **(c-g)** we note that coefficients $\binom{\nu}{\lambda}$ coincide with $\hat{\zeta}(\lambda, \nu)$ in the notation of [BO, §2.2] (Proposition 2.5 and the discussion afterwards). Its inverse (denoted

$\hat{\mu}(\lambda, \nu)$ is given by

$$\sum_{\lambda \xrightarrow{i} \nu} \binom{\nu = \nu}{\lambda \xrightarrow{i} \nu} \chi(\text{coker}(i)), \quad [\text{BO}, \text{Claim 2.6}].$$

We get

$$(6) \quad \mathbf{g}_\nu = \sum_{\lambda \leq \nu} \sum_i \binom{\nu = \nu}{\lambda \xrightarrow{i} \nu} \chi(\text{coker}(i)) \mathbf{c}_\lambda.$$

The following Lemma shows that many of the terms in (6) vanish.

Lemma 3.1. *If $\mathfrak{p}y \neq (0)$ then $C(y)$ is contractible, in particular, $\chi(y) = 0$.*

Proof. Denote $C = C(y)$, and $D \subseteq C$ the subcomplex spanned by the subset of vertices $\{x \mid \mathfrak{p}y \subseteq x \subset y\} \subseteq C_0$. D is a cone over the vertex $\mathfrak{p}y$, hence contractible. The function

$$\varphi : \mathcal{L}(y) \rightarrow \mathcal{L}(y), \quad \varphi(x) = x + \mathfrak{p}y$$

extends to a retraction $\varphi_* : C \rightarrow D$. The function

$$\Psi : C \times [0, 1] \rightarrow C \quad \Psi(c, t) = (1 - t)c + t\varphi_*(c)$$

establishes a deformation retract from C to D . Therefore C is contractible. \square

The value of $\chi(y)$ depends only on the isomorphism type of y hence we shall use the notation $\chi(\lambda)$ for $\lambda \in \Lambda_n$. The vanishing of $\chi(\lambda)$ when \mathfrak{o}_λ is not a vector space will be written in short as $\chi(\lambda) = 0$ if $\mathfrak{p}\lambda = 0$.

3.2. Explicit calculation. It will be useful to use another set of coordinates for elements in Λ_n^ℓ , obtained by transposed diagrams $\lambda' = (\lambda'_j)$, defined by $\lambda'_j = |\{i : \lambda_i \geq j\}|$. The module-theoretic interpretation of the λ'_j 's is given in [Mac, II.1(1.4)]. For every partition ξ , let $n(\xi) = \sum (i - 1)\xi_i$ and $|\xi| = \sum \xi_i$. Let $q = |\mathfrak{o}/\mathfrak{p}| = |k_F|$, and for $m, n \in \mathbb{N}$ set

$$\begin{aligned} [n]_q &= 1 - q^{-n}, & [0]_q &= 1, \\ [n]_q! &= [n]_q [n-1]_q \cdots [1]_q, & \begin{bmatrix} m \\ n \end{bmatrix}_q &= \frac{[n]_q!}{[m]_q! [n-m]_q!}. \end{aligned}$$

The subscript q will be occasionally omitted from the notation.

Proposition 3.2.

$$\mathbf{g}_\lambda = \sum_{\{\nu \mid \phi \geq \nu \geq \lambda \geq \mathfrak{p}\nu\}} (-1)^{|\nu| - |\lambda|} q^{n(\nu) - n(\lambda)} \prod_{i \geq 1} \begin{bmatrix} \nu'_i - \nu'_{i+1} \\ \nu'_i - \lambda'_i \end{bmatrix} \cdot \mathbf{c}_\nu,$$

Proof. Lemma 3.1 combined with a well known formula [Rot, CR] giving the Euler characteristic of the Tits building associated to $\text{GL}_n(k_F)$, gives for every module type λ ,

$$(7) \quad \chi(\lambda) = \begin{cases} 0 & \mathfrak{p}\lambda \neq 0 \\ (-1)^{\dim_{k_F}(\lambda)} q^{\binom{\dim_{k_F}(\lambda)}{2}} & \mathfrak{p}\lambda = 0 \end{cases}$$

whereas the number

$$\sum_{\{\lambda \xrightarrow{i} \nu \mid \mathfrak{p}\text{coker}(i) = 0\}} \binom{\nu = \nu}{\lambda \xrightarrow{i} \nu}$$

is exactly the Hall coefficient $G_{\lambda, 1(|\nu| - |\lambda|)}^\nu$. The latter is explicitly computed in [Mac, II.4]. \square

4. TRANSITION MATRIX: CELLULAR TO IDEMPOTENTS

4.1. Abstract description. Let E be a fixed \mathfrak{o} -module of type $\Phi = \ell^n$. We say that two submodules are *transversal* if their intersection is zero. Let κ, λ and μ be types of modules and let x_μ and x_κ be two transversal submodules of E of types μ and κ respectively. Let

- $[\mu \prec \dot{\lambda} \cap \kappa]_\Phi$ be the number of submodules of type λ in F which contain x_μ and are transversal to x_κ .
- $[\mu \prec \dot{\lambda}]_\Phi$ be the number of submodules of F of type λ which contain a given submodule of type μ (in the above notation this is $[\mu \prec \dot{\lambda} \cap 0]_\Phi$).

Note that both $[\mu \prec \dot{\lambda} \cap \kappa]_\Phi$ and $[\mu \prec \dot{\lambda}]_\Phi$ are well defined by Lemma 2.3.

Since the cellular structure agrees with the idempotent decomposition, we already know by Corollary 3 that there exist a lower triangular matrix $A_{\lambda\kappa}$ such that the relation (c-e) above holds. We have already seen that the transition matrix (c-g) from the geometric basis to the cellular basis depends only on geometric invariants of the lattice of submodules in a very simple way. This is also the case for the cellular-idempotents transition matrix.

Theorem 4. $A_{\lambda\kappa} = [\kappa \prec \dot{\lambda}]_\Phi [\lambda \prec \dot{\phi} \cap \kappa]_\Phi$.

Our strategy is to analyze the multiplication in the algebra with respect to the cellular basis. Let $B_{\lambda\mu}^\nu$ be multiplication coefficients with respect to the cellular basis

$$(8) \quad \mathbf{c}_\lambda \cdot \mathbf{c}_\kappa = \sum_{\nu \leq \lambda \wedge \kappa} B_{\lambda\kappa}^\nu \mathbf{c}_\nu.$$

Observe that $B_{\lambda\kappa}^\kappa = A_{\lambda\kappa}$ for $\kappa \leq \lambda$. The following lemma follows immediately from [BO, Lemma 3.6].

Lemma 4.1.

- (1) $T_{\phi \succ \kappa} T_{\kappa \succ \nu} = [\nu \prec \dot{\kappa}]_\Phi T_{\phi \succ \nu}$.
- (2) $T_{\nu \prec \kappa} T_{\kappa \prec \phi} = [\nu \prec \dot{\kappa}]_\Phi T_{\nu \prec \phi}$.

Substituting $\mathbf{c}_\eta = T_{\phi \succ \eta} T_{\eta \prec \phi}$ in equation (8) and using Lemma 4.1 gives (assume $\kappa \leq \lambda$):

$$(9) \quad \begin{aligned} & T_{\phi \succ \lambda} \left(T_{\lambda \prec \phi} T_{\phi \succ \kappa} \right) T_{\kappa \prec \phi} \\ &= T_{\phi \succ \lambda} \left(\sum_{\nu \leq \kappa} \frac{B_{\lambda\kappa}^\nu}{[\nu \prec \dot{\lambda}]_\Phi [\nu \prec \dot{\kappa}]_\Phi} T_{\lambda \succ \nu} T_{\nu \prec \kappa} \right) T_{\kappa \prec \phi} \end{aligned}$$

Let $\mathcal{F}_\lambda^\bullet = \text{Im}(T_{\lambda \prec \phi})$. Observe that the set of maps $\Delta = \Delta_{\lambda\kappa} = \{T_{\lambda \succ \nu} T_{\nu \prec \kappa}\}_{\nu \leq \lambda \wedge \kappa}$ when restricted to $\mathcal{F}_\kappa^\bullet$, forms a basis for G_ℓ -maps $\mathcal{F}_\kappa^\bullet \rightarrow \mathcal{F}_\lambda^\bullet$. Indeed, these maps are independent after being composed with $T_{\phi \succ \lambda}$ on the left and $T_{\kappa \prec \phi}$ on the right and using lemma 4.1. We get the following identity on $\mathcal{F}_\lambda^\bullet$:

$$(10) \quad \begin{aligned} T_{\lambda \prec \phi} T_{\phi \succ \kappa} &= \sum_{\nu \leq \kappa} \frac{B_{\lambda\kappa}^\nu}{[\nu \prec \dot{\lambda}]_\Phi [\nu \prec \dot{\kappa}]_\Phi} T_{\lambda \succ \nu} T_{\nu \prec \kappa} \\ &= \frac{A_{\lambda\kappa}}{[\kappa \prec \dot{\lambda}]_\Phi} T_{\lambda \succ \kappa} + \{\text{terms with } \nu < \kappa\} \end{aligned}$$

Denote the coefficient of an operator S w.r.t to a basis element $D \in \Delta$ by $\langle S, D \rangle_\Delta$.

Definition 4.2. A triple $\kappa \leq \lambda \leq \eta$ is called good if

$$\langle T_{\lambda \prec \eta} T_{\eta \succ \kappa}, T_{\lambda \succ \kappa} \rangle_{\Delta} = [\lambda \prec \eta \dot{\cap} \kappa]_{\Phi}$$

Combining Definition 4.2 with (10) we see that Theorem 4 is equivalent to:

Theorem 5. $\kappa \leq \lambda \leq \phi$ is a good triple.

We would like to take a small pause and explain the strategy which we undertake. The idea is to show that it is enough to find a path connecting λ and ϕ in the segment $[\lambda, \phi]$ which can be paved with good triples, and then exhibit such path. More precisely, we follow three steps:

- (1) Given $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_r \leq \phi$ such that $\kappa \leq \lambda_{i-1} \leq \lambda_i$ is good for all $1 \leq i \leq r$ and also $\kappa \leq \lambda_r \leq \phi$ is good, implies that so is $\kappa \leq \lambda_0 \leq \phi$.
- (2) $\kappa \leq \lambda_1 \leq \lambda_2$ is good whenever λ_2 covers λ_1 (i.e. $[\lambda_1, \lambda_2] = \{\lambda_1, \lambda_2\}$) and has the same rank.
- (3) $\kappa \leq \epsilon \leq \phi$ is good whenever ϵ and ϕ are symmetric.

Note that a Jordan-Hölder sequence of types from λ to a symmetric type ϵ of the same rank gives an appropriate path: $\lambda = \lambda_0 \leq \dots \leq \lambda_r = \epsilon \leq \phi$.

Step 1

Lemma 4.3. If $\kappa \leq \lambda \leq \theta$ and $\kappa \leq \theta \leq \phi$ are good so is $\kappa \leq \lambda \leq \phi$.

Proof.

$$\begin{aligned} \langle T_{\lambda \prec \phi} T_{\phi \succ \kappa}, T_{\lambda \succ \kappa} \rangle_{\Delta} &= \frac{1}{[\lambda \prec \dot{\theta}]_{\phi}} \langle (T_{\lambda \prec \theta} T_{\theta \prec \phi}) T_{\phi \succ \kappa}, T_{\lambda \succ \kappa} \rangle_{\Delta} \\ &= \frac{[\theta \prec \dot{\phi} \dot{\cap} \kappa]}{[\lambda \prec \dot{\theta}]_{\phi}} \langle T_{\lambda \prec \theta} (T_{\theta \prec \kappa} + \text{lower terms}), T_{\lambda \succ \kappa} \rangle_{\Delta} \\ &= \frac{[\theta \prec \dot{\phi} \dot{\cap} \kappa]}{[\lambda \prec \dot{\theta}]_{\phi}} \langle T_{\lambda \prec \theta} T_{\theta \prec \kappa}, T_{\lambda \succ \kappa} \rangle_{\Delta} \\ &= \frac{[\theta \prec \dot{\phi} \dot{\cap} \kappa][\lambda \prec \dot{\theta}]_{\kappa}}{[\lambda \prec \dot{\theta}]_{\phi}} = \frac{[\lambda \prec \dot{\theta} \prec \dot{\phi} \dot{\cap} \mu]}{[\lambda \prec \dot{\theta}]_{\phi}} = [\lambda \prec \dot{\phi} \dot{\cap} \kappa]_{\Phi} \end{aligned}$$

□

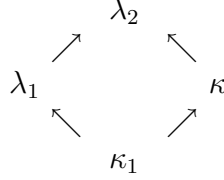
Using this lemma r times completes the proof of step 1. Note that the lemma can be used successively only from the 'top' to the 'bottom'.

Step 2

Let λ_1 and λ_2 be types of same rank and assume that λ_2 covers λ_1 . Let $\kappa \leq \lambda_1$. We want to show that $\langle T_{\lambda_1 \prec \lambda_2} T_{\lambda_2 \succ \kappa}, T_{\lambda_1 \succ \kappa} \rangle_{\Theta} = [\lambda_1 \prec \dot{\lambda}_2 \dot{\cap} \kappa]$. However, the assumption that λ_1 and λ_2 have the same rank guarantees that any module of type λ_2 containing a module of type λ_1 which is transversal w.r.t. a module of type κ inherits this transversality. Hence, the requirement to avoid κ is redundant and $[\lambda_1 \prec \dot{\lambda}_2 \dot{\cap} \kappa] = [\lambda_1 \prec \dot{\lambda}_2]$. We start by expanding the product $T_{\lambda_1 \prec \lambda_2} T_{\lambda_2 \succ \kappa}$:

$$(11) \quad T_{\lambda_1 \prec \lambda_2} T_{\lambda_2 \succ \kappa} = a_{\lambda_1} T_{\lambda_1 \hookrightarrow \lambda_1 \hookleftarrow \kappa} + a_{\lambda_2} T_{\lambda_1 \hookrightarrow \lambda_2 \hookleftarrow \kappa}$$

The assumptions on λ_1 and λ_2 assures that no other terms appear in (11). Evidently $T_{\lambda_1 \succ \kappa} = T_{\lambda_1 \hookrightarrow \lambda_1 \hookleftarrow \kappa}$ and $a_{\lambda_1} = [\lambda_1 \prec \lambda_2]$. We are therefore reduced to showing that the Δ expansion of the second term in (11) does not contain a multiple of $T_{\lambda_1 \succ \kappa}$. This is accomplished by claim 4.4. Let κ_1 be the unique type which can (possibly) complete a cartesian diagram (see the first part of claim A.3 in appendix A):



Claim 4.4. $(T_{\lambda_1 \hookrightarrow \lambda_2 \hookleftarrow \kappa})|_{\mathcal{F}_\kappa^\bullet} \in \mathbb{C} \cdot (T_{\lambda_1 \succ \kappa_1} T_{\kappa_1 \prec \kappa})|_{\mathcal{F}_\kappa^\bullet}$.

Proof. We shall prove the equivalent statement:

$$T_{\phi \succ \lambda_1} T_{\lambda_1 \hookrightarrow \lambda_2 \hookleftarrow \kappa} \in \mathbb{C} \cdot T_{\phi \succ \lambda_1} T_{\lambda_1 \succ \kappa_1} T_{\kappa_1 \prec \kappa} \quad (= \mathbb{C} \cdot T_{\phi \succ \kappa_1} T_{\kappa_1 \prec \kappa} \text{ by lemma 4.1})$$

We begin by applying the r.h.s. to a cyclic element $\delta_{x_0} \in \mathcal{F}_\kappa$:

$$\begin{aligned}
 [T_{\phi \succ \kappa_1} T_{\kappa_1 \prec \kappa} \delta_{x_0}](y_0) &= \sum_{\substack{z \subset y_0 \\ \tau(z) = \kappa_1}} [T_{\kappa_1 \prec \kappa} \delta_{x_0}](z) = \sum_{\substack{z \subset y_0 \\ \tau(z) = \kappa_1}} \sum_{\substack{x \supset z \\ \tau(x) = \phi}} \delta_{x_0}(x) \\
 &= \sum_{\substack{z \subset y_0 \\ \tau(z) = \kappa_1}} \mathbf{1}_{\{z | z \subset x_0, \tau(z) = \kappa_1\}} = |\{z | z \subset y_0 \cap x_0, \tau(z) = \kappa_1\}| \\
 &= \begin{cases} \binom{\kappa}{\kappa_1} & \text{if } y_0 \supset x_0 \\ 1 & \text{if } \tau(y_0 \cap x_0) = \kappa_1 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Let w_0 , x_0 and y_0 be submodules of F such that $w_0 \subset x_0 \cap y_0$ with types $\tau(w_0) = \kappa_1$, $\tau(x_0) = \kappa$ and $\tau(y_0) = \phi$. Define:

$$B = B_{x_0, y_0, w_0} = \{z | z \subset y_0, \tau(z) = \lambda_1, \tau(z + x_0) = \lambda_2, z \cap x_0 = w_0\}$$

By the second part of claim A.3:

$$B = \{z | w_0 \subset z \subset y_0, \tau(z) = \lambda_1, \tau(\mathbf{p}z + \mathbf{p}x_0) = \mathbf{p}\lambda_2, \mathbf{p}z \cap \mathbf{p}x_0 = \mathbf{p}w_0\}$$

This description of B together with the symmetricity of y_0 implies that $|B|$ depends only on the types of $\mathbf{p}x_0$ and w_0 , denote it by $b_{\mathbf{p}\kappa, \kappa_1}$. Applying the l.h.s. to δ_{x_0} yields:

$$\begin{aligned}
 [T_{f \succ \lambda_1} T_{\lambda_1 \hookrightarrow \lambda_2 \hookleftarrow \kappa} \delta_{x_0}](y_0) &= \sum_{\substack{z \subset y_0 \\ \tau(z) = \lambda_1}} [T_{\lambda_1 \hookrightarrow \lambda_2 \hookleftarrow \kappa} \delta_{x_0}](z) = \sum_{\substack{z \subset y_0 \\ \tau(z) = \lambda_1}} \sum_{\substack{x \supset z \\ \tau(z+x) = \lambda_2}} \delta_{x_0}(x) \\
 &= |\{z | z \subset y_0, \tau(z) = \lambda_1, \tau(z + x_0) = \lambda_2\}| \\
 &= \begin{cases} \binom{\kappa}{\kappa_1} b_{\mathbf{p}\kappa, \kappa_1} & \text{if } y_0 \supset x_0 \\ b_{\mathbf{p}\kappa, \kappa_1} & \text{if } \tau(y_0 \cap x_0) = \kappa_1 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Hence, the two operators differ by a constant. □

Step 3

We should prove that the triple $\kappa \leq \epsilon \leq \phi$ is good when both ϵ and ϕ are symmetric types. This is precisely the assertion of [BO, Theorem 6]. In particular see relation (8) in the proof. The only delicate point which deserves a remark, is the duality axiom which is used in the proof and should be justified. Indeed, in a module of type κ , the number of submodules of type α equals to the number of submodules of co-type α (cf. [Mac, II.1]).

4.2. Explicit calculation. Recall the notations of §3.2. For partitions $\lambda, \nu \in \cup \Lambda_j$ let

$$\langle \lambda, \nu \rangle = \sum \lambda_i \nu_i.$$

Claim 4.5.

$$\begin{aligned} (1) \quad \binom{\nu}{\lambda} &= q^{\langle \nu' - \lambda', \lambda' \rangle} \prod_{i \geq 1} \begin{bmatrix} \nu'_i - \lambda'_{i+1} \\ \nu'_i - \lambda'_i \end{bmatrix}. \\ (2) \quad [\lambda \prec \dot{\nu}]_{\phi} &= q^{\langle \phi' - \nu', \nu' - \lambda' \rangle} \begin{bmatrix} \phi'_1 - \lambda'_1 \\ \phi'_1 - \nu'_1 \end{bmatrix} \prod_{i \geq 1} \begin{bmatrix} \nu'_i - \lambda'_{i+1} \\ \nu'_i - \nu'_{i+1} \end{bmatrix}. \\ (3) \quad [\dot{\nu} \cap \lambda]_{\Phi} &= q^{\langle \nu', \lambda'_1 \rangle} \binom{\ell^{n-\lambda'_1}}{\nu}. \\ (4) \quad [\nu \prec \dot{\phi} \cap \lambda]_{\Phi} &= q^{\langle \Phi' - \phi', \phi' - \nu' \rangle} \begin{bmatrix} \Phi'_1 - \nu'_1 - \lambda'_1 \\ \phi'_1 - \nu'_1 \end{bmatrix}. \end{aligned}$$

Proof. A basic quantity, which all other quantities are scaled to, is the cardinality of $\text{Hom}(\mathfrak{o}_{\lambda}, \mathfrak{o}_{\nu})$ which is denoted and computed by

$$\text{hom}(\lambda, \nu) = q^{\langle \lambda', \nu' \rangle}$$

The subset of all injective morphisms will be denoted $\text{Hom}^{1-1}(\mathfrak{o}_{\lambda}, \mathfrak{o}_{\nu})$ and we will use

$$\text{hom}^{1-1}(\lambda, \nu) = |\text{Hom}^{1-1}(\mathfrak{o}_{\lambda}, \mathfrak{o}_{\nu})|$$

For a given type λ , $\text{hom}^{1-1}(\lambda, \lambda)$ is computed in [Mac, II.1] where it is denoted $a_{\lambda}(q)$. A similar computation shows that

$$\text{hom}^{1-1}(\lambda, \nu) = \prod_{i \geq 1} \frac{[\nu'_i - \lambda'_{i+1}]!}{[\nu'_i - \lambda'_i]!} \text{hom}(\lambda, \nu).$$

Observe that the map

$$\text{Hom}^{1-1}(\mathfrak{o}_{\lambda}, \mathfrak{o}_{\nu}) \rightarrow \text{Gr}(\lambda, \mathfrak{o}_{\nu}), \quad \psi \mapsto \text{Im}(\psi)$$

is $\text{hom}^{1-1}(\lambda, \lambda)$ to one, thus

$$\binom{\nu}{\lambda} = \frac{\text{hom}^{1-1}(\lambda, \nu)}{\text{hom}^{1-1}(\lambda, \lambda)},$$

and (1) follows. Given an \mathfrak{o} -module f of type ϕ , counting in two ways the size of the set

$$\{x, y < f \mid x < y, \tau(x) = \lambda, \tau(y) = \nu\}$$

gives

$$\binom{\phi}{\lambda} [\lambda \prec \dot{\nu}]_{\phi} = \binom{\phi}{\nu} \binom{\nu}{\lambda},$$

which combined with (1) proves (2).

In order to prove (3) we need a little preparation. Let E be a module of type Φ . Let $z < E$ be a fixed module of type λ . Denote by E_1 a module of type $\ell^{\lambda'_1}$ containing z . Observe that E_1 is a direct summand of E . Fix a complimentary direct summand to E_1 such that $E = E_1 \oplus E_2$. Denote the corresponding projections by p_1 and p_2 . Assume that a type ν is given. Let

$$X = \{x < E \mid \tau(x) = \nu, x \cap z = 0\},$$

in particular $[\dot{\nu} \cap \lambda] = |X|$. Observe that the map

$$\begin{aligned} \text{Hom}(\mathfrak{o}_\nu, E_1) \oplus \text{Hom}^{1-1}(\mathfrak{o}_\nu, E_2) &\rightarrow X, \\ (\psi_1, \psi_2) &\mapsto \text{Im}(\psi_1 + \psi_2) \end{aligned}$$

is $\text{hom}^{1-1}(\nu, \nu)$ to one. This map is indeed into X as

$$\text{Im}(\psi_1 \oplus \psi_2) \cap z \subset \text{Im}(\psi_1 \oplus \psi_2) \cap E_1 \simeq \text{Ker}(\psi_2) = (0).$$

It is onto X as for a given $\psi \in \text{Hom}^{1-1}(\mathfrak{o}_\nu, E)$, with $\text{Im}(\psi) \in X$ we have $\text{Ker}(p_2 \circ \psi) \simeq \text{Im}(\psi) \cap E_1$ which is (0) , as the \mathfrak{p} -torsion of E_1 is equal to the \mathfrak{p} -torsion of z . We get that

$$[\dot{\nu} \cap \lambda] = \binom{\ell^{n-\lambda'_1}}{\nu} \text{hom}(\nu, \ell^{\lambda'_1}),$$

which proves (3). Counting in two ways the size of the set

$$\{x, y < E \mid x < y, y \cap z = 0, \tau(x) = \nu, \tau(y) = \phi\}$$

gives

$$[\dot{\phi} \cap \lambda]_\Phi \binom{\phi}{\nu} = [\dot{\nu} \cap \lambda]_\Phi [\nu \prec \dot{\phi} \cap \lambda]_\Phi$$

which combined with (3) proves (4). □

Corollary 6.

$$A_{\lambda\kappa} = q^{\langle \phi' - \lambda', \Phi' - \phi' + \lambda' - \kappa' \rangle} \begin{bmatrix} \phi'_1 - \kappa'_1 \\ \phi'_1 - \lambda'_1 \end{bmatrix} \begin{bmatrix} \Phi'_1 - \lambda'_1 - \kappa'_1 \\ \phi'_1 - \lambda'_1 \end{bmatrix} \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \kappa'_{i+1} \\ \lambda'_i - \lambda'_{i+1} \end{bmatrix}.$$

5. FROM THE FINITE GRASSMANNIANS TO $\mathcal{S}(\text{Gr}(m, n, F))$

Recall that I_ℓ denotes the kernel of the reduction of $\text{GL}_n(\mathfrak{o})$ modulo \mathfrak{p}^ℓ . The group $\text{GL}_n(\mathfrak{o})$, being the inverse limit of the finite groups G_{ℓ^n} , enjoys the property that each of its continuous irreducible complex representations has a *level*. That is, the first nonnegative integer ℓ such that $I_{\ell+1}$ acts trivially. It follows that there exist a natural filtration

$$(0) \subset \mathcal{S}(\text{Gr}(m, n, F))^{I_1} \subset \mathcal{S}(\text{Gr}(m, n, F))^{I_2} \subset \cdots \subset \mathcal{S}(\text{Gr}(m, n, F)).$$

The ℓ -th term in this filtration consists of all the irreducible components of the representation which have level at most $\ell - 1$, and thus can be regarded as a representation of G_{ℓ^n} . Denote this representation by $\bar{\rho}_{\ell^m}$. Since each irreducible constituent is captured by some $\bar{\rho}_{\ell^m}$, we get

$$(12) \quad \mathcal{S}(\text{Gr}(m, n, F)) \simeq \varinjlim \bar{\rho}_{\ell^m}$$

as $\text{GL}_n(\mathfrak{o})$ representations.

Lemma 5.1. $(\bar{\rho}_{\ell^m}, \mathcal{S}(\text{Gr}(m, n, F))^{I_\ell}) \simeq (\rho_{\ell^m}, \mathcal{F}_{\ell^m})$.

Proof. Follows from the natural identification

$$I_\ell \backslash \text{Gr}(m, n, F) \simeq I_\ell \backslash \text{GL}_n(F) / P_m(F) \simeq \text{GL}_n(\mathfrak{o}_\ell) / P_m(\mathfrak{o}_\ell) \simeq X_{\ell^m},$$

where P_m is the appropriate parabolic group, and the isomorphism

$$\mathcal{S}(\text{Gr}(m, n, F))^{I_\ell} \simeq \mathbb{C}(I_\ell \backslash \text{Gr}(m, n, F)).$$

□

Consequently, combining (12) with Lemma 5.1 proves Claim 2.1. Note that the isomorphism in Lemma 5.1 is algebraic, and that there are two different inner products on ρ_{ℓ^m} and on $\bar{\rho}_{\ell^m}$, arising from the counting measure on X_{ℓ^m} or the projection of the (probability) Haar measure from $\text{GL}_n(\mathfrak{o})$ to $I_\ell \backslash \text{Gr}(m, n, F)$, respectively. Throughout the finite analysis we kept the former inner product, while at this stage of transferring the results to the infinite Grassmann representation, we should keep track of the appropriate normalization.

The whole study included here hinges on the profinite nature of the ring of integers \mathfrak{o} , and hence of all groups, spaces and algebras defined over it, summarized by

Groups	$\text{GL}_n(\mathfrak{o}) \simeq \varprojlim \text{GL}_n(\mathfrak{o}_\ell)$
Spaces	$\text{Gr}(m, n, F) \simeq \varprojlim \text{Gr}(m, n, \mathfrak{o}_\ell)$
Representations	$\mathcal{S}(\text{Gr}(m, n, F)) \simeq \varinjlim \mathcal{F}_{\ell^m}$
Algebras	$\mathcal{H}_m \simeq \varinjlim \text{End}_{\text{GL}_n(\mathfrak{o})}(\mathcal{F}_{\ell^m})$

and explained in detail below.

5.1. Lifting the finite spaces, algebras and functions. Let $\pi_\ell : X_{\ell^m} \rightarrow X_{(\ell-1)^m}$ be the natural quotient maps. As $\text{GL}_n(\mathfrak{o})$ -spaces we have

$$\text{Gr}(m, n, F) \simeq \varprojlim X_{\ell^m}.$$

Using the identification in Claim 2.2, the maps π_ℓ descent to maps

$$\pi_\ell : \Lambda_m^\ell \rightarrow \Lambda_m^{\ell-1}, \quad \tau(\mathfrak{o}_\lambda) \mapsto \tau(\mathfrak{o}_\lambda / \mathfrak{p}^{\ell-1} \mathfrak{o}_\lambda)$$

which are easily described in transposed coordinates by

$$(\lambda'_1, \dots, \lambda'_{\ell-1}, \lambda'_\ell) \mapsto (\lambda'_1, \dots, \lambda'_{\ell-1}).$$

Consider the set $\coprod_{\ell \geq 0} \Lambda_m^\ell$, and endow it with a graph structure by connecting each $\lambda \in \Lambda_m^\ell$ with its image $\pi_\ell(\lambda) \in \Lambda_m^{\ell-1}$. This graph is a rooted tree, the root being the empty partition in Λ_m^0 . The inverse limit $\varprojlim \Lambda_m^\ell$ can be identified with the space of ends of this tree. The obvious sections $\Lambda_m^{\ell-1} \rightarrow \Lambda_m^\ell$ given by $(\lambda'_1, \dots, \lambda'_{\ell-1}) \mapsto (\lambda'_1, \dots, \lambda'_{\ell-1}, 0)$ give at the limit an imbedding of Λ_m in $\varprojlim \Lambda_m^\ell$. Thus, Λ_m can be identified with an open and dense subset of $\varprojlim \Lambda_m^\ell$. Let $\bar{\mathbb{N}}$ stand for the one point compactification of \mathbb{N} . Λ_m is naturally imbedded in $\bar{\mathbb{N}}^m$. We denote by $\bar{\Lambda}_m$ its closure in $\bar{\mathbb{N}}^m$. It is easily seen that $\bar{\Lambda}_m$ can be identified with $\varprojlim \Lambda_m^\ell$. We summarize this discussion by

Proposition 5.2. $\text{Gr}(m, n, F) \times_{\text{GL}_n(\mathfrak{o})} \text{Gr}(m, n, F) \simeq \bar{\Lambda}_m \simeq \varprojlim \Lambda_m^\ell$.

Proof. The only nontrivial issue left to address is the fact that the first identification is also topological. The topology on the l.h.s is the quotient topology. The quotient map from $\text{Gr}(m, n, F) \times \text{Gr}(m, n, F)$ to $X_{\ell^m} \times X_{\ell^m}$ is continuous and $\text{GL}_n(\mathfrak{o})$ -equivariant. The limit map becomes continuous and well-defined on the quotient. □

Remark 5.3.

- (1) As topological spaces, $\bar{\Lambda}_m \setminus \Lambda_m \simeq \bar{\Lambda}_{m-1}$, thus $\coprod_{i=0}^m \Lambda_i$ is a stratification of $\bar{\Lambda}_m$.
- (2) $\text{Gr}(m, n, F)$ carries a $\text{GL}_n(\mathfrak{o})$ -invariant measure. Consequently, also does $\bar{\Lambda}_m$. Apparently, Λ_m is of full measure inside $\bar{\Lambda}_m$. This measure is computed in [Onn, §2.2].

The maps π_ℓ give rise to inclusions of the (finite dimensional) spaces $i_\ell : L^2(X_{(\ell-1)^m}) \rightarrow L^2(X_{\ell^m})$, where the notation $\mathcal{F}(X)$ is replaced by $L^2(X)$ to emphasize that the inner product structure is induced from the Haar measure, rather than the counting measure. The adjoint transformation, i_ℓ^* , is the orthogonal projection on the $I_{\ell-1}$ invariants. In the limit we get the vector space of Bruhat-Schwartz (=locally constant) functions:

$$\mathcal{S}(\text{Gr}(m, n, F)) \simeq \varinjlim L^2(\text{Gr}(m, n, F))^{I_\ell} \simeq \varinjlim L^2(X_{\ell^m}).$$

$L^2(\text{Gr}(m, n, F))$ is the completion $\mathcal{S}(\text{Gr}(m, n, F))$, or alternatively, the direct limit in the category of Hilbert spaces. The inclusions i_ℓ also give embeddings of the Hecke algebras $\mathcal{H}_{\ell^m} = \text{End}_{G_{\ell^m}}(L^2(X_{\ell^m}))$ given by

$$\mathcal{H}_{(\ell-1)^m} \rightarrow \mathcal{H}_{\ell^m}, \quad h \mapsto i_\ell \circ h \circ i_\ell^*.$$

Recall by (3) that as vector spaces $\mathcal{H}_{\ell^m} \simeq \mathcal{F}(\Lambda_m^\ell)$, and that under this isomorphism the operator $\mathbf{g}_\lambda \in \mathcal{H}_{\ell^m}$ ($\lambda \in \Lambda_m^\ell$) corresponds to the delta function $\delta_\lambda^\ell \in \mathcal{F}(\Lambda_m^\ell)$ supported on λ .

Claim 5.4. *If $\lambda'_{\ell-1} = 0$ then $i_\ell \circ \delta_\lambda^{\ell-1} \circ i_\ell^* = \delta_\lambda^\ell$.*

Proof. The condition $\lambda'_{\ell-1} = 0$ implies that $\pi_\ell^{-1}(\lambda)$ is the singleton $\{\lambda\} \subset \Lambda_m$, and the claim follows. \square

It follows that the image of δ_λ^ℓ inside $\varinjlim \mathcal{H}_{\ell^m}$ stabilizes for ℓ large enough, thus determining an element $\delta_\lambda = \varinjlim \delta_\lambda^\ell$, where δ_λ is the delta function supported at λ , viewing λ as an element of $\bar{\Lambda}_m$ via $\Lambda_m \subset \bar{\Lambda}_m$. The identifications of Proposition 5.2 give three ways to look at

$$\mathcal{H}_m = \text{End}_{\text{GL}_n(\mathfrak{o})}(\mathcal{S}(\text{Gr}(, m, n, F))),$$

namely,

$$\mathcal{H}_m \simeq \mathcal{S}(\text{Gr}(m, n, F) \times_{\text{GL}_n(\mathfrak{o})} \text{Gr}(m, n, F)) \simeq \mathcal{S}(\bar{\Lambda}_m) \simeq \varinjlim \mathcal{H}_{\ell^m}.$$

Here $\mathcal{S}(\bar{\Lambda}_m)$ is the space of locally constant functions on $\bar{\Lambda}_m$. The limit algebra structure obviously coincides with the operator algebra structure of $\mathcal{S}(\bar{\Lambda}_m)$, arising when viewing its elements as convolution operators on $\mathcal{S}(\text{Gr}(m, n, F))$. Denote by $\mathcal{F}(\Lambda_m)$ the space of finitely supported functions on Λ_m . As Λ_m is discrete in $\bar{\Lambda}_m$, $\mathcal{F}(\Lambda_m)$ is imbedded in $\mathcal{S}(\bar{\Lambda}_m)$. As Λ_m is dense in $\bar{\Lambda}_m$, $\mathcal{F}(\Lambda_m)$ is dense in $\mathcal{S}(\bar{\Lambda}_m)$ too. Consequently, the algebraic structure of $\mathcal{S}(\bar{\Lambda}_m)$ is determined by $\mathcal{F}(\Lambda_m) = \text{Span}\{\delta_\lambda : \lambda \in \Lambda_m\}$.

5.2. Transition matrices. We are finally in a position to collect the pieces, and write down explicitly the transition matrix between the delta functions basis of \mathcal{H}_m and the idempotents of \mathcal{H}_m . In order to do that we introduce an intermediate basis, which is the limit of the (normalized image in \mathcal{H}_m) of the finite levels cellular bases. For $\lambda \in \Lambda_m$ define

$$\bar{\mathbf{c}}_\lambda = \varinjlim_\ell \frac{\mathbf{c}_\lambda}{\binom{\Phi}{\phi}},$$

$\bar{\mathbf{g}}_\lambda$ the operator which corresponds to δ_λ , and $\bar{\mathbf{e}}_\lambda$ the image of \mathbf{e}_λ in \mathcal{H}_m . Combining the results of §3-4 with the above discussion gives

Theorem 7.

$$\begin{aligned}
\bar{\mathbf{g}}_\lambda &= \sum_{\kappa \geq \lambda \geq \mathfrak{p}\kappa} \hat{\mu}(\lambda, \kappa) \bar{\mathbf{c}}_\kappa \\
&= \sum_{\kappa \geq \lambda \geq \mathfrak{p}\kappa} (-1)^{|\nu| - |\lambda|} q^{n(\nu) - n(\lambda)} \prod \left[\begin{matrix} \nu'_i - \nu'_{i+1} \\ \nu'_i - \lambda'_i \end{matrix} \right] \bar{\mathbf{c}}_\kappa \\
\bar{\mathbf{c}}_\kappa &= \sum_{\nu \leq \kappa} \frac{[\nu \prec \dot{\kappa}]_\phi [\kappa \prec \dot{\phi} \upharpoonright \nu]_\Phi}{(\Phi)} \bar{\mathbf{e}}_\nu \\
&= \sum_{\nu \leq \kappa} q^{-(n-2m)|\kappa| - m|\nu| - \langle \kappa', \kappa' - \nu' \rangle} \frac{\left[\begin{matrix} m - \nu'_1 \\ m - \kappa'_1 \end{matrix} \right] \left[\begin{matrix} n - \kappa'_1 - \nu'_1 \\ m - \kappa'_1 \end{matrix} \right]}{\left[\begin{matrix} n \\ m \end{matrix} \right]} \prod \left[\begin{matrix} \kappa'_i - \nu'_{i+1} \\ \kappa'_i - \nu'_i \end{matrix} \right] \bar{\mathbf{e}}_\nu
\end{aligned}$$

Which together give the desired $(\bar{\mathbf{g}} - \bar{\mathbf{e}})$ transition matrix.

6. GRASSMANN REPRESENTATION OVER NONSYMMETRIC MODULES AND OPEN PROBLEMS

6.1. Complexity of the representations \mathcal{F}_λ . The focus of this paper is on the representations \mathcal{F}_λ with $\lambda = \ell^m$. They enjoy the property that their decomposition into irreducible constituents is of combinatorial nature, in particular, independent of the ring \mathfrak{o} . For arbitrary (non-rectangular) types there is a strong dependence on the ring, and a highly non trivial problem is

Problem 1. *Decompose \mathcal{F}_λ into irreducible constituents for any $\lambda \in \Lambda_m$.*

The nontriviality of the problem is demonstrated in the next proposition. Let B_j denote the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathfrak{o}_j)$, that is, the stabilizer of a full flag of \mathfrak{o}_j -free submodules in \mathfrak{o}_j^n .

Proposition 6.1. *If all parts of $\lambda \in \Lambda_n$ are pairwise unequal with smallest part $\lambda_n = j$ and largest part $\lambda_1 = \ell$, then the G_{ℓ^n} -representation \mathcal{F}_λ contains $\mathrm{Ind}_{B_j}^{G_{j^n}}(1)$, where the action of G_{ℓ^n} on the latter is via reduction modulo \mathfrak{p}^j .*

To wit the complexity of $\mathrm{Ind}_{B_j}^{G_{j^n}}(1)$, the reader is referred to [CN] in which the case $n = 3$ is studied. Though we know very little about these arbitrary Grassmannians, they can be used to pin down the irreducible representations which occur in the Grassmann representation studied in the current paper: the essence of the labeling in (1) of §1 comes from the following theorem.

Theorem 8. [BO] *There exist a family $\{\mathcal{U}_\lambda^F \mid \lambda \in \Lambda_m\}$ of irreducible representations of $\mathrm{GL}_n(\mathfrak{o})$ such that*

- (1) $\mathcal{S}(\mathrm{Gr}(m, n, F)) = \bigoplus_{\lambda \in \Lambda_m} \mathcal{U}_\lambda^F$.
- (2) $\langle \mathcal{U}_\lambda^F, \mathcal{F}_\mu \rangle = |\{\lambda \hookrightarrow \mu\}|$. *I.e., the multiplicity of \mathcal{U}_λ^F in \mathcal{F}_μ is the number of nonequivalent embeddings of a module of type λ in a module of type μ .*

In particular \mathcal{U}_λ^F occurs both in $\mathcal{S}(\mathrm{Gr}(m, n, F))$ and in \mathcal{F}_λ with multiplicity one, and does not occur in \mathcal{F}_μ for $\lambda \not\leq \mu$.

6.2. Dimensions of \mathcal{U}_λ^F . The dimensions of the representations \mathcal{U}_λ^F were computed in [OS], using a sophisticated and heavy computational machinery of degenerations of certain generalized quantum dimensions formulae to actual dimensions of the \mathcal{U}_λ^F 's. They are given by the following formula [OS, §4.3].

$$(13) \quad \dim_{\mathbb{C}}(\mathcal{U}_\lambda^F) = t^{-(n-2m+1)|\lambda|-2(\rho,\lambda)} \left[\begin{matrix} m \\ \partial\lambda' \end{matrix} \right]_t \frac{(t^{n-\lambda'_1-\lambda'_2+2}; t)_{\lambda'_1+\lambda'_2} (1-t^{n-2\lambda'_1+1})}{(t^{m-\lambda'_1+1}; t)_{\lambda'_1} (1-t^{n+1})}$$

for $\lambda \in \Lambda_n$.

Here $\partial\lambda' = (\lambda'_j - \lambda'_{j+1})_{j \geq 0}$, $\rho = (n-1, n-2, \dots, 0)$, $(a; t)_j = \prod_{i=0}^{j-1} (1-t^i a)$ and $t = |\mathfrak{o}/\mathfrak{p}|^{-1}$. We used t instead of q to match the notation of [OS].

Problem 2. Compute the dimensions of \mathcal{U}_λ^F ($\lambda \in \Lambda_m$) directly.

6.3. Heisenberg-like relations on the complete lattice of submodules. Let \mathcal{L} denote the lattice of submodules in \mathfrak{o}_ℓ^n and let $\Lambda = \Lambda_n^\ell$. Let $\mathcal{F}(\mathcal{L}) = \bigoplus_{\lambda \leq \ell^n} \mathcal{F}_\lambda$ stand for complex valued functions on \mathcal{L} . For $x, y \in \mathcal{L}$ we use the notation $y \succ x$ whenever y covers x (i.e. y/x is simple). Define the following 'lowering' and 'raising' operators on $\mathcal{F}(\mathcal{L})$:

$$\begin{aligned} \mathbf{D}^\flat : \mathcal{F}(\mathcal{L}) &\rightarrow \mathcal{F}(\mathcal{L}) & \mathbf{D}^\sharp : \mathcal{F}(\mathcal{L}) &\rightarrow \mathcal{F}(\mathcal{L}) \\ \mathbf{D}^\flat f(x) &= \sum_{y \succ x} f(y) & \mathbf{D}^\sharp f(x) &= \sum_{y \prec x} f(y). \end{aligned}$$

Observe that \mathbf{D}^\flat and \mathbf{D}^\sharp are adjoints. Indeed, this follows from $\mathbf{D}^\flat = \sum_{\mu \prec \lambda} T_{\mu \prec \lambda}$ and $\mathbf{D}^\sharp = \sum_{\mu \succ \lambda} T_{\mu \succ \lambda}$.

Proposition 6.2. $(\mathbf{D}^\flat \mathbf{D}^\sharp - \mathbf{D}^\sharp \mathbf{D}^\flat)|_{\mathcal{F}_\lambda} = b_\lambda \cdot \text{Id}_{\mathcal{F}_\lambda}$

Proof. Using the definition of \mathbf{D}^\flat and \mathbf{D}^\sharp , we need to show that

$$\sum_{y \succ x} \sum_{z \prec y} h(z) - \sum_{y \prec x} \sum_{z \succ y} h(z) = b_\lambda \cdot h(x) \quad \forall x \in X_\lambda$$

For any subset $\Sigma \subset \Lambda$ let $\sharp(\Sigma)$ be the set of types which covers types from Σ and $\flat(\Sigma)$ the set of types which are covered by types from Σ . For $\lambda \in \Lambda$ let $\natural(\lambda) = \flat\sharp(\lambda) = \sharp\flat(\lambda)$.

First, we note that any $y \neq x$ from $\tau^{-1}(\natural(\lambda))$, appears exactly once in each of the summands on the left hand side. Indeed, there is exactly one submodule (their join) which covers both of them, and exactly one submodule (their meet) which is covered by both of them. Hence such pairs do not contribute to the left hand side.

Second, $y = x \in \tau^{-1}(\natural(\lambda))$ appears u_λ times in the first summand, and l_λ times in the second summand, where for a fixed $z_0 \in \tau^{-1}(\lambda)$:

$$u_\lambda = |\{y | y \succ z_0\}| \quad l_\lambda = |\{y | y \prec z_0\}|$$

It follows that $b_\lambda = u_\lambda - l_\lambda$. □

The scalars b_λ can be easily computed. They are given by

$$b_\lambda = \frac{q^{n-\text{rk}\lambda} - q^{\text{rk}\lambda}}{q - 1}$$

which follows from

$$l_\lambda = |\{y|y \leq z_0\}| = |\mathbb{P}_{k_F}^{\text{rk}(\lambda)-1}| = \frac{q^{\text{rk}(\lambda)} - 1}{q - 1}$$

$$u_\lambda = |\{y|y \geq z_0\}| = |\mathbb{P}_{k_F}^{n-\text{rk}(\lambda)-1}| = \frac{q^{n-\text{rk}(\lambda)} - 1}{q - 1}.$$

6.4. More questions regarding the Hecke algebra \mathcal{H}_{ℓ^m} . The transition matrix $(\mathbf{c}-\mathbf{e})$ is given explicitly by a combinatorial data (Theorem 4). Examples imply that this should also be the case for the transition matrix $(\mathbf{e}-\mathbf{c})$, and it would be interesting to find such interpretation.

Problem 3. *Invert the relation $(\mathbf{c}-\mathbf{e})$.*

As mentioned in the introduction the case $\ell = 1$ is well studied [Dun, Del]. In *loco citato* the set X_{1^m} is studied as an association scheme (the q -Johnson scheme). The graph structure is defined by: two points $x, y \in X_{1^m}$ are connected with an edge if $x \cap y$ is of codimension one in each of them. The Laplacian Δ on this graph, defined by $\Delta h(x) = \sum_{x \sim y} h(y)$ (for $h \in \mathcal{F}_{1^m}$), is nothing but the operator \mathbf{g}_{1^m-1} . It turns out that Δ generates the Hecke algebra, i.e. $\mathcal{H}_{1^m} = \mathbb{C}[\Delta]$. The following is a conjectural generalization of this fact.

Problem 4. *Prove that $\{\mathbf{g}_{(\ell^m-1, j^1)} \mid j = 0, \dots, \ell-1\}$ generate \mathcal{H}_{ℓ^m} as an algebra.*

6.5. Other algebraic groups. The case $\ell = 1$ admits generalizations to other algebraic groups, see e.g. [Sta]. It would be interesting to generalize these further to other classical groups. For example

Problem 5. *Study the natural Grassmann representation of the symplectic group arising from its action on Lagrangian subspaces of ℓ^{2n} .*

APPENDIX A. LOCAL RINGS AND DISCRETE VALUATION RINGS

In this appendix we prove some claims regarding local rings. All modules under consideration are assumed to be of finite rank. Let R be a local ring with a maximal ideal \mathfrak{p} and let x, y be R -modules. Let $\bar{x} = x/\mathfrak{p}x$. By Nakayama's lemma $x \rightarrow y$ is onto if and only if the induced map $\bar{x} \rightarrow \bar{y}$ is onto. Equivalently $\text{rk}(x) = \dim x/\mathfrak{p}x$.

Claim A.1. *Let R be a local ring with maximal ideal \mathfrak{p} . Let z be an R -module and x, y two submodules of z . Then:*

$$\text{rk}(x + y) = \text{rk}(x) + \text{rk}(y) - \text{rk}(x \cap y) + \dim \left(\frac{\mathfrak{p}x \cap \mathfrak{p}y}{\mathfrak{p}(x \cap y)} \right)$$

Proof. Using the equalities $\mathfrak{p}(x+y) = \mathfrak{p}x + \mathfrak{p}y$ and $\mathfrak{p}(x \oplus y) = \mathfrak{p}x \oplus \mathfrak{p}y$, we obtain a commutative diagram with exact columns and rows (14) and exact sequence (15):

$$(14) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathfrak{p}x \cap \mathfrak{p}y & \rightarrow & \mathfrak{p}x \oplus \mathfrak{p}y & \rightarrow & \mathfrak{p}x + \mathfrak{p}y \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & x \cap y & \rightarrow & x \oplus y & \rightarrow & x + y \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{x \cap y}{\mathfrak{p}x \cap \mathfrak{p}y} & \dashrightarrow & \frac{x \oplus y}{\mathfrak{p}x \oplus \mathfrak{p}y} & \dashrightarrow & \frac{x + y}{\mathfrak{p}(x + y)} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

(the exactness of the dashed row follows from the obvious exactness of the other rows and columns)

$$(15) \quad 0 \rightarrow \frac{\mathfrak{p}x \cap \mathfrak{p}y}{\mathfrak{p}(x \cap y)} \rightarrow \frac{x \cap y}{\mathfrak{p}(x \cap y)} \rightarrow \frac{x \cap y}{\mathfrak{p}x \cap \mathfrak{p}y} \rightarrow 0$$

and obtain:

$$\begin{aligned} \text{rk}(x) + \text{rk}(y) = \text{rk}(x \oplus y) &\stackrel{(14)}{=} \text{rk}(x + y) + \dim \frac{x \cap y}{\mathfrak{p}x \cap \mathfrak{p}y} \\ &\stackrel{(15)}{=} \text{rk}(x + y) + \text{rk}(x \cap y) - \dim \left(\frac{\mathfrak{p}x \cap \mathfrak{p}y}{\mathfrak{p}(x \cap y)} \right) \end{aligned}$$

□

Claim A.2. *Let R be a local ring with maximal ideal \mathfrak{p} . Let z be an R -module and x, y two submodules of z . Then:*

$$\text{rk}(x) = \text{rk}(x + y) \iff \text{rk}(x \cap y) = \text{rk}(y) \text{ and } \mathfrak{p}x \cap \mathfrak{p}y = \mathfrak{p}(x \cap y)$$

Proof. (\Leftarrow) clear. (\Rightarrow) if $\text{rk}(x) = \text{rk}(x + y)$ we have that the two non-negative terms $\text{rk}(y) - \text{rk}(x \cap y)$ and $\dim \left(\frac{\mathfrak{p}x \cap \mathfrak{p}y}{\mathfrak{p}(x \cap y)} \right)$ must sum up to zero. □

We now specialize to the situation to which we apply these assertions.

Claim A.3. *Let \mathfrak{o} be a discrete valuation ring with maximal ideal \mathfrak{p} . Let z be a finite \mathfrak{o} -module and x, y two submodules of z . Assume that $\text{rk}(x + y) = \text{rk}(x)$ and $x + y$ covers x .*

- (1) $\tau(x \cap y)$ depends only on $\tau(x)$, $\tau(y)$ and $\tau(x + y)$.
- (2) $\tau(x + y) = \lambda \iff \tau(\mathfrak{p}x + \mathfrak{p}y) = \mathfrak{p}\lambda$, $\mathfrak{p}x \cap \mathfrak{p}y = \mathfrak{p}(x \cap y)$ and $\mathfrak{p}(x \cap y) \subsetneq \mathfrak{p}y$.

Proof. One immediately verifies that for a module w , $\tau(w) = \lambda \iff \dim \bar{w} = \lambda_1 = \text{rk}(\lambda)$ and $\tau(\mathfrak{p}w) = \mathfrak{p}\lambda$.

- (1) There exist a unique i such that $\mathfrak{p}^i(x + y)/\mathfrak{p}^i x \neq (0)$. By the isomorphism $\mathfrak{p}^i(x + y)/\mathfrak{p}^i x \simeq \mathfrak{p}^i y/\mathfrak{p}^i(x \cap y)$ it is also the unique i for which $\mathfrak{p}^i y/\mathfrak{p}^i(x \cap y) \neq (0)$.
- (2) (\Rightarrow) Assume $\tau(x + y) = \lambda$. Clearly $\tau(\mathfrak{p}x + \mathfrak{p}y) = \mathfrak{p}\lambda$, and by Claim A.2 also $\mathfrak{p}x \cap \mathfrak{p}y = \mathfrak{p}(x \cap y)$. Since $x \cap y \subsetneq y$ and have the same rank, the dimensions of $\overline{x \cap y}$ and \bar{y} must be the same and $\mathfrak{p}(x \cap y) \subsetneq \mathfrak{p}y$.

(\Leftarrow) The data on the right together with claim A.2 implies that $\tau(\mathfrak{p}(x + y)) = \mathfrak{p}\lambda$ and $\text{rk}(x + y) = \text{rk}(x)$. This implies that $\tau(x + y) = \lambda$. □

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